Int J Theor Phys (2010) 49: 2891–2901
DOI 10.1007/s10773-010-0480-5

The Generalized Thermo Jaynes–Cummings Model and Its Energy Spectrum

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Received: 28 June 2010 / Accepted: 24 August 2010 / Published online: 7 September 2010

Abstract By analyzing the characteristics of thermo field dynamics, we present a new generalized thermo Jaynes-Cummings (TJC) model at finite temperature including the thermal effects and expatiate on its physical explanation. By virtue of Lewis-Riesenfeld invariant method, we obtain its exact eigenenergy spectrum and the explicit expressions for the evolution operator of the wavefunctions. In addition, we evaluate various expectation values of physical quantities and find the pseudo-invariant eigen-operator for the generalized TJC Hamiltonian.

Keywords Supersymmetric generators · Eigenenergy spectrum · Thermo Jaynes-Cummings model

1 Introduction

The interaction of a two-level atom with a single mode of the electromagnetic field, described by the Jaynes-Cummings (JC) model [1], is one of the simplest and most fundamental quantum systems. It is typically realized in cavity QED experiments [2, 3] in different frequency regimes and configurations. Over the last many years much attention has been focused on its various nonlinear extensions in quantum optics. Such models are shown to exhibit interesting nonclassical effects, such as the collapse and revivals of Rabi oscillations [4, 5], antibunched light and squeezing [6, 7], inversionless light amplification [8], etc. The extensions of the JC model are mainly along two directions. On the one hand, one

Work supported by the National Natural Science Foundation of China under Grant Nos. 10775097 and 10874174.

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still considers a two-level atom and a single-mode quantized light field but taking multiphoton processes into account [9, 10]. On the other hand, one can consider a three-level atom and multimode quantized cavity fields, and then turns this system into an effective two-level problem by the adiabatic elimination approximation [4] or perturbation transformation method [11]. To our knowledge, all these JC models are discussed at zero-temperature environment. But in nature most systems are immersed in a thermal reservoir and thermal excitation and de-excitation processes are influenced by the exchange of energy between the system and the reservoir. In order to tackle this problem, Barnett and Knight [12] have first proposed thermo Jaynes-Cummings (TJC) Model. Recently, Fan's group [13] has also presented another thermo Jaynes-Cummings model at finite temperature by analyzing the characteristics of Takahashi-Umezawa thermo field dynamics [14–16].

For thermo field dynamics, the original Fock state should be doubled and a fictitious Hilbert space $\tilde{H}$ (tilde-conjugate field) should be introduced, i.e., every annihilation operator $\alpha(\vec{k})$ with momentum $\hbar\vec{k}$ acting on the original Hilbert space H has a counterpart $\tilde{\alpha}(\vec{k})$ acting on the fictitious Hilbert space $\tilde{H}$, which satisfies $[\tilde{\alpha}(\vec{k}), \tilde{\alpha}^\dagger(\vec{k})] = [\alpha(\vec{k}), \alpha^\dagger(\vec{k})] = 1$. Correspondingly, every number state $|m\rangle = \alpha^{\dagger m}/\sqrt{m!}|0\rangle$ in H is accompanied by a tilde state $|\tilde{m}\rangle = \tilde{\alpha}^{\dagger m}/\sqrt{m!}|\tilde{0}\rangle$ in $\tilde{H}$. At finite temperature T the thermal vacuum state is defined by

$$|0(\beta)\rangle = S(\theta)\left|0\tilde{0}\right\rangle = \operatorname{sech}\theta \exp\left[\alpha^\dagger(\vec{k})\tilde{\alpha}^\dagger(\vec{k})\tanh\theta\right]|0,\tilde{0}\rangle \tag{1}$$

where $\beta = 1/\kappa T$ (κ is the Boltzmann constant), $S(\theta)$ is called the thermal operator

$$S(\theta) = \exp\left[\theta\left(\alpha^\dagger(\vec{k})\tilde{\alpha}^\dagger(\vec{k}) - \alpha(\vec{k})\tilde{\alpha}(\vec{k})\right)\right], \tag{2}$$

and the vacuum state $|0,\tilde{0}\rangle$ is annihilated by either α or $\tilde{\alpha}$. The average photon number in the thermal vacuum state obeys the Bose-Einstein distribution,

$$\langle 0(\beta)|\,\alpha^\dagger(\vec{k})\alpha(\vec{k})\,|0(\beta)\rangle = \sinh^2\theta = \frac{1}{(e^{\hbar\omega\beta} - 1)}. \tag{3}$$

In the present paper, we motivate our pioneering work [13] and extend the TJC model to the generalized case at finite temperature including the thermal effects, which is expressed as

$$\mathcal{H} = \omega h + \frac{1}{2}\Omega\sigma_z + \lambda\left[\sigma_+\left(\sqrt{\frac{\alpha - \tilde{\alpha}^\dagger}{\alpha^\dagger - \tilde{\alpha}}}\sqrt{h}\right)^n + \left(\sqrt{h}\sqrt{\frac{\alpha^\dagger - \tilde{\alpha}}{\alpha - \tilde{\alpha}^\dagger}}\right)^n\sigma_-\right] \tag{4}$$

where $h \equiv \alpha^\dagger\alpha - \tilde{\alpha}^\dagger\tilde{\alpha}$, ω and Ω are the thermal field frequency and the atom transition frequency, respectively, λ is the coupling constant, and the Pauli matrices σ_z, $\sigma_\pm$ represent the two-level atomic system.

$$\sigma_+\left(\sqrt{\left(\alpha - \tilde{\alpha}^\dagger\right)/\left(\alpha^\dagger - \tilde{\alpha}\right)}\sqrt{h}\right)^n \quad \text{and} \quad \left(\sqrt{h}\sqrt{\left(\alpha^\dagger - \tilde{\alpha}\right)/\left(\alpha - \tilde{\alpha}^\dagger\right)}\right)^n\sigma_-$$

describe the interaction between the thermal photon field and a two-level atom. Our main purpose is to derive the exact and explicit eigenenergy spectrum for the generalized TJC model and obtain various expectation values of physical quantities for the given initial states.

The paper is arranged as follows. In Sect. 2, based on the thermal excitation representation [17], we further present the physical explanation involved in (4) in more detail. In

Sect. 3, we redescribe the generalized TJC Hamiltonian in (4) via the supersymmetric generators. In Sect. 4, by virtue of the Lewis-Riesenfeld invariant method [18] extensively employed to investigate various quantum evolution problems [19], we deal with the generalized TJC model and obtain its exact eigenenergy spectrum and the explicit expression for the evolution operator of the wavefunctions. In addition, we shall evaluate various expectation values of physical quantities, such as the thermo excitation number and the atomic inversion operator. In Sect. 5, according to the theory of pseudo-invariant eigen-operator [21], we search for the pseudo-invariant eigen-operator of the generalized TJC model, which directly leads to the energy-level gap of this system.

2 Physical Explanation of $\mathcal{H}$

In this section, let us further expatiate on the physical explanation of the Hamiltonian $\mathcal{H}$ in (4). For this purpose, we first recall excitation and de-excitation operator and the corresponding representation in thermo field dynamics.

As Takahashi-Umezawa pointed out [14], the Hamiltonian of the total system including the effects of the reservoir should be weakly equal to

$$H = \hbar \int d^3k\omega(\vec{k}, \beta)h(\vec{k}, \beta), \tag{5}$$

where

$$h(\vec{k}, \beta) \equiv \alpha^\dagger(\vec{k}, \beta)\alpha(\vec{k}, \beta) - \tilde{\alpha}^\dagger(\vec{k}, \beta)\tilde{\alpha}(\vec{k}, \beta), \tag{6}$$

the creation operator $\alpha^\dagger(\vec{k}, \beta)$ describes the excitation of an additional quantum with positive energy $\hbar\omega$ and momentum $\hbar\vec{k}$. Under the thermal transformation $S(\theta)$,

$$\alpha(\vec{k}, \beta) = S(\theta)\alpha(\vec{k})S^{-1}(\theta) = \alpha(\vec{k})\cosh\theta - \tilde{\alpha}^\dagger(\vec{k})\sinh\theta, \tag{7}$$

$$\tilde{\alpha}(\vec{k}, \beta) = S(\theta)\tilde{\alpha}(\vec{k})S^{-1}(\theta) = \tilde{\alpha}(\vec{k})\cosh\theta - \alpha^\dagger(\vec{k})\sinh\theta. \tag{8}$$

From (6) and (7) and (8), it is clear that

$$h = \alpha^\dagger(\vec{k})\alpha(\vec{k}) - \tilde{\alpha}^\dagger(\vec{k})\tilde{\alpha}(\vec{k}) \equiv \alpha^\dagger\alpha - \tilde{\alpha}^\dagger\tilde{\alpha}, \tag{9}$$

which implies that h is invariant under the thermal transformation. Here the notation $\vec{k}$ are omitted for convenience.

In order to give a physical explanation of $\mathcal{H}$, following the Fan's work [25], we introduce the following state

$$\|e, m\rangle = \left(\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\right)^e |m, \tilde{m}\rangle\,, \tag{10}$$

where e is an integer and $|m, \tilde{m}\rangle = \alpha^{\dagger m}\widetilde{\alpha}^{\dagger m}/m!|0, \tilde{0}\rangle$ is the number state spanning the whole system (original + tilde). For $e \geq 0$, one obtains the explicit expansion form of $\|e, m\rangle$ in Fock space [25]

$$\|e,m\rangle = \Gamma\left(\frac{e}{2}+1\right)\sum_{m'=0}^{\infty}\sum_{k=0}^{\min(m',m)}\sqrt{\frac{m'!}{(m'+e)!}} \times \binom{-\frac{e}{2}}{m'-k}\binom{\frac{e}{2}}{m-k}\binom{\frac{e}{2}+k}{k}\left|m',\widetilde{m+e}\right\rangle. \tag{11}$$

Due to the commutative relations

$$\left[h,(\alpha-\widetilde{\alpha}^{\dagger})(\alpha^{\dagger}-\widetilde{\alpha})\right]=0, \tag{12}$$

$$\left[h,\alpha-\widetilde{\alpha}^{\dagger}\right]=-\left(\alpha-\widetilde{\alpha}^{\dagger}\right), \tag{13}$$

and

$$\left[h,\alpha^{\dagger}-\widetilde{\alpha}\right]=\alpha^{\dagger}-\widetilde{\alpha}, \tag{14}$$

one easily obtains that

$$\left[h,\sqrt{\frac{\alpha^{\dagger}-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^{\dagger}}}\right]=\sqrt{\frac{\alpha^{\dagger}-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^{\dagger}}} \tag{15}$$

and

$$\left[h,\sqrt{\frac{\alpha-\widetilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\widetilde{\alpha}}}\right]=-\sqrt{\frac{\alpha-\widetilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\widetilde{\alpha}}}. \tag{16}$$

Considering (15) and (16) and $h|m,\tilde{m}\rangle = 0$, we can prove that

$$h\,\|e,m\rangle = \left[h,\left(\sqrt{\frac{\alpha^{\dagger}-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^{\dagger}}}\right)^{e}\right]|m,\tilde{m}\rangle = e\,\|e,m\rangle\,, \tag{17}$$

which means that $\|e,m\rangle$ is the eigenstate of the operator h with the eigenvalue being e. From (10), it is easily proved that

$$\sqrt{\frac{\alpha-\tilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\tilde{\alpha}}}\,\|e,m\rangle = \|e-1,m\rangle\,, \tag{18}$$

and

$$\sqrt{\frac{\alpha^{\dagger}-\tilde{\alpha}}{\alpha-\tilde{\alpha}^{\dagger}}}\,\|e,m\rangle = \|e+1,m\rangle\,, \tag{19}$$

which indicates that $\sqrt{\frac{\alpha-\widetilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\widetilde{\alpha}}}$ and $\sqrt{\frac{\alpha^{\dagger}-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^{\dagger}}}$ plays the role of lowering (de-excitation) and ascending (excitation), respectively, when operating them on the state $\|e,m\rangle$. Further, it is then followed that

$$\left(\sqrt{\frac{\alpha-\tilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\tilde{\alpha}}}\sqrt{h}\right)^{n}\|e,m\rangle = \sqrt{e\,(e-1)\cdots(e-n+1)}\,\|e-n,m\rangle\,, \tag{20}$$

and

$$\left(\sqrt{h}\sqrt{\frac{\alpha^{\dagger}-\tilde{\alpha}}{\alpha-\tilde{\alpha}^{\dagger}}}\right)^{n}\|e,m\rangle=\sqrt{(e+n)\cdots(e+2)(e+1)}\,\|e+n,m\rangle. \tag{21}$$

Moreover, it is not difficult to prove that $\|e,m\rangle$ has a complete and orthogonal set

$$\langle e',m'\|e,m\rangle=\delta_{e'e}\delta_{m'm}, \qquad \sum_{e=-\infty}^{\infty}\sum_{m=0}^{\infty}\|e,m\rangle\langle e,m\|=1, \tag{22}$$

which makes it a useful representation.

Based on the state $\|e,m\rangle$, now we can afford an explanation of $\mathcal{H}$ as follows. Because e is the eigenvalue of h from (17), we name e the excitation number of the total system's energy and $\|e,m\rangle$ the thermal excitation representation. It is seen from (18) and (19) that the lowering operator $\sqrt{\frac{\alpha-\tilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\tilde{\alpha}}}$ and ascending operator $\sqrt{\frac{\alpha^{\dagger}-\tilde{\alpha}}{\alpha-\tilde{\alpha}^{\dagger}}}$ of thermal photon field are embodied in the thermal state $\|e,m\rangle$. The term $\sigma_{+}(\sqrt{\frac{\alpha-\tilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\tilde{\alpha}}}\sqrt{h})^{n}$ involved in (4) denotes that the thermal photon of thermal field is de-excitation and the atom is excited from the lower level to the upper level. In this process, the interaction is proportional to $\sqrt{h}$ and n is the thermal photon multiple. While the term $(\sqrt{h}\sqrt{\frac{\alpha^{\dagger}-\tilde{\alpha}}{\alpha-\tilde{\alpha}^{\dagger}}})^{n}\sigma_{-}$ denotes the opposite process, namely, the thermal photon of thermal field is excitation and the atom is excited from the upper level to the lower level. Thus, the established generalized thermo Jaynes-Cummings model should be considered in the space $\|e,m\rangle$.

Finally, we mention that when finite temperature case reduces to zero temperature case, i.e., $T\to 0$, from (1) and (3) $\sinh\theta\to 0$, $\cosh\theta\to 1$ and $|0(\beta)\rangle\to|0,\tilde{0}\rangle$. In this case, the Hamiltonian in (4) recovers to that of the multiphoton process [9]

$$\widetilde{H}=\omega_{0}\alpha^{\dagger}\alpha+\frac{1}{2}\Omega\sigma_{z}+\chi\left(\sigma_{+}\alpha^{n}+\alpha^{\dagger n}\sigma_{-}\right)$$

where χ denotes the multiphoton atom-field coupling coefficient, n is the photon multiple and ω_0 is the frequency of the cavity mode.

3 Supersymmetric Generators of $\mathcal{H}$

In order to deal with the generalized TJC, we first redescribe the generalized TJC Hamiltonian via the supersymmetric operators and denote the supersymmetric generators as

$$Q\equiv\left(\sqrt{h}\sqrt{\frac{\alpha^{\dagger}-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^{\dagger}}}\right)^{n}\sigma_{-}=\begin{pmatrix}0 & 0\\ (\sqrt{h}\sqrt{\frac{\alpha^{\dagger}-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^{\dagger}}})^{n} & 0\end{pmatrix}, \tag{23}$$

$$Q^{\dagger}\equiv\sigma_{+}\left(\sqrt{\frac{\alpha-\widetilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\widetilde{\alpha}}}\sqrt{h}\right)^{n}=\begin{pmatrix}0 & \left(\sqrt{\frac{\alpha-\widetilde{\alpha}^{\dagger}}{\alpha^{\dagger}-\widetilde{\alpha}}}\sqrt{h}\right)^{n}\\ 0 & 0\end{pmatrix}, \tag{24}$$

$$N\equiv h+\frac{n}{2}\sigma_{z}, \tag{25}$$

and

$$N' \equiv \begin{pmatrix} (h+1)(h+2)\cdots(h+n) & 0 \\ 0 & (h-n+1)\cdots(h-1)h \end{pmatrix}. \tag{26}$$

It is easily seen that N, N', Q, $Q^\dagger$ form the supersymmetric generators and have Lie superalgebra properties, i.e.,

$$[Q, \sigma_z] = 2Q, \qquad [Q^\dagger, \sigma_z] = -2Q^\dagger \tag{27}$$

$$Q^2 = Q^{\dagger 2} = 0, \qquad \{Q, \sigma_z\} = \{Q^\dagger, \sigma_z\} = 0 \tag{28}$$

$$[N, Q] = [N, Q^\dagger] = \left[N', Q\right] = [N', Q^\dagger] = 0, \tag{29}$$

and

$$[Q^\dagger, Q] = N'\sigma_z, \qquad (Q^\dagger - Q)^2 = -N', \qquad \{Q, Q^\dagger\} = N', \tag{30}$$

where $\{\}$ denotes the anticommutation relations.

In terms of the above generators, we can rewrite the Hamiltonian (4) as

$$\mathcal{H} = \omega N + \frac{1}{2}\varpi\sigma_z + \lambda\left(Q^\dagger + Q\right), \tag{31}$$

where $\varpi = \Omega - n\omega$ is called the detuning quantity.

4 Eigenenergy Spectrum of $\mathcal{H}$

Based on the above section, we shall use the Lewis-Riesenfeld invariant method to solve the generalized TJC model involved in (31) and obtain its exact eigenenergy spectrum and the explicit expressions for the evolution operator of the wavefunctions.

Begin with, we summarize some related results for the Lewis-Riesenfeld invariant method [18–20]. For a given time-dependent Hamiltonian $H(t)$, it can be supposed that there is a Hermitian invariant operator $I(t)$ which satisfies ($\hbar = 1$)

$$\frac{\partial I}{\partial t} - i[I, H] = 0. \tag{32}$$

If there exists a complete orthogonal set of the eigenstates $|\phi(t)\rangle_j$ of $I(t)$ with eigenvalues β_j, i.e.,

$$I\,|\phi(t)\rangle_j = \beta_j\,|\phi(t)\rangle_j\,, \tag{33}$$

then we can express the exact solution $|\Psi(t)\rangle$ and the time evolution operator $U(t)$ for the system in terms of instantaneous eigenstates of $I(t)$ as

$$|\Psi(t)\rangle = \sum_j C_j \exp\left[i\varphi_j(t)\right]|\phi(t)\rangle_j \tag{34}$$

and

$$U(t,0) = \sum_j \exp\left[i\varphi_j(t)\right]|\phi(t)\rangle_{jj}\langle\phi(0)| \tag{35}$$

with the phase factor

$$\varphi_j(t) = \int_0^t dt'_j \langle \phi(t') | \left[i\frac{\partial}{\partial t'} - H(t') \right] | \phi(t') \rangle_j , \tag{36}$$

and

$$C_j = {}_j\langle \phi(0) | \Psi(0) \rangle \tag{37}$$

Thus the problems of solving time-dependent Schrödinger equation are turned into finding the instantaneous eigenspectra of the invariant chosen adequately. Obviously, the invariant method is also applicable for the cases with time-independent Hamiltonian.

Since the Lewis-Riesenfeld invariant method can only be applied to study of such a system for which there exists a closed quasialgebra, it is easy to see from (30) that there does not exist such an algebra for the system. However, applying $\sigma_z|\pm\rangle = \pm|\pm\rangle$ and (17), we get that the eigenvalue equation of N' is

$$N' \begin{pmatrix} \|e, m\rangle \\ \|e+n, m\rangle \end{pmatrix} = \mathcal{N}_n \begin{pmatrix} \|e, m\rangle \\ \|e+n, m\rangle \end{pmatrix}, \tag{38}$$

with the eigenvalue $\mathcal{N}_n = (e+n)\cdots(e+2)(e+1)$, where

$$\begin{pmatrix} \|e, m\rangle \\ 0 \end{pmatrix} \equiv \|e, m\rangle \otimes |+\rangle = \|e, m\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{39}$$

and

$$\begin{pmatrix} 0 \\ \|e+n, m\rangle \end{pmatrix} \equiv \|e+n, m\rangle \otimes |-\rangle = \|e+n, m\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{40}$$

In the eigenspace of the operator N', one finds from (30) that

$$[Q^\dagger, Q] = \mathcal{N}_n \sigma_z, \qquad (Q^\dagger - Q)^2 = -\mathcal{N}_n, \qquad \{Q, Q^\dagger\} = \mathcal{N}_n. \tag{41}$$

By considering the closure property of (27)–(29) and (41), the Hermitian invariant $I(t)$ for the generalized TJC in (31) can be constructed as

$$I = f\left(Q^\dagger + Q\right) + g\sigma_z, \tag{42}$$

where f, g are real quantities with time-independent to be determined. Substituting (42) and (31) into (32) and equating the coefficients of Q, $Q^\dagger$ and σ_z both sides, we can derive the following equations

$$\frac{df}{dt} = i\left(-\varpi f + 2g\lambda\right), \qquad \frac{dg}{dt} = 0 \tag{43}$$

which yields

$$f = \frac{2g\lambda}{\varpi}. \tag{44}$$

Considering (23) and (24), we write the matrix representation of I and suppose its eigenequation described as

$$\begin{pmatrix} g & f\left(\sqrt{\frac{\alpha-\widetilde{\alpha}^\dagger}{\alpha^\dagger-\widetilde{\alpha}}}\sqrt{h}\right)^n \\ f\left(\sqrt{h}\sqrt{\frac{\alpha^\dagger-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^\dagger}}\right)^n & -g \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} x \\ y \end{pmatrix} \tag{45}$$

whose four types of solutions is

$$x=\cos\frac{\gamma}{2}\left\|e,m\right\rangle,\qquad y=\sin\frac{\gamma}{2}\left\|e+n,m\right\rangle, \tag{46}$$

$$x=-\sin\frac{\gamma}{2}\left\|e,m\right\rangle,\qquad y=\cos\frac{\gamma}{2}\left\|e+n,m\right\rangle, \tag{47}$$

$$x=0,\qquad y=\left\|e,m\right\rangle,\quad e=0,1,\ldots,(n-1), \tag{48}$$

and

$$x=\left\|e,m\right\rangle,\qquad y=0,\quad e=-n,\ldots-2,-1. \tag{49}$$

Note that we have used the property that the eigenvalues β are real quantities.

By substituting (46)–(49) into (45), then the eigenvalues β and normalized eigenfuntions $|\phi\rangle$ of I can be obtained as follows

$$\beta_1=g\sqrt{\Delta}/\varpi,\qquad |\phi\rangle_1=\begin{pmatrix}\cos\frac{\gamma}{2}\left\|e,m\right\rangle\\ \sin\frac{\gamma}{2}\left\|e+n,m\right\rangle\end{pmatrix}, \tag{50}$$

$$\beta_2=-g\sqrt{\Delta}/\varpi,\qquad |\phi\rangle_2=\begin{pmatrix}-\sin\frac{\gamma}{2}\left\|e,m\right\rangle\\ \cos\frac{\gamma}{2}\left\|e+n,m\right\rangle\end{pmatrix}, \tag{51}$$

$$\beta_3=-g,\qquad |\phi\rangle_{3e}=\begin{pmatrix}0\\ \left\|e,m\right\rangle\end{pmatrix},\quad e=0,1,\ldots,(n-1), \tag{52}$$

and

$$\beta_4=g,\qquad |\phi\rangle_{4e}=\begin{pmatrix}\left\|e,m\right\rangle\\ 0\end{pmatrix},\quad e=-n,\ldots,-2,-1, \tag{53}$$

where $\Delta\equiv\varpi^2+4\lambda^2\mathcal{N}_n$, and $\gamma=\tan^{-1}(\frac{2\lambda\sqrt{\mathcal{N}_n}}{\varpi})$.

In the following section, we proceed to calculate the time evolution of state vector of the system and its evolution operator. We may easily obtain the time evolution eigenvectors $|\Phi\rangle_j$ with the geometric phase factor $\varphi_j(t)$, and the corresponding eigenvalues E_j as follows

$$\begin{aligned}&|\Phi(t)\rangle_1=\exp\left[i\varphi_1(t)\right]|\phi\rangle_1,\\ &\varphi_1(t)=-{}_1\langle\Phi|\mathcal{H}|\Phi\rangle_1\,t=-E_1t=-\left(\omega e+\frac{\varpi}{2\cos\gamma}+\frac{n\omega}{2}\right)t,\end{aligned} \tag{54}$$

$$\begin{aligned}&|\Phi(t)\rangle_2=\exp\left[i\varphi_2(t)\right]|\phi\rangle_2,\\ &\varphi_2(t)=-{}_2\langle\Phi|\mathcal{H}|\Phi\rangle_2\,t=-E_2t=-\left(\omega e-\frac{\varpi}{2\cos\gamma}+\frac{n\omega}{2}\right)t,\end{aligned} \tag{55}$$

$$\begin{aligned}&|\Phi\rangle_{3e}=\exp\left[i\varphi_{3e}(t)\right]|\phi\rangle_{3e},\\ &\varphi_{3e}(t)=-{}_{3e}\langle\Phi|\mathcal{H}|\Phi\rangle_{3e}\,t=-E_{3e}t\\ &\quad=-\left(\omega e+\frac{\varpi}{2\cos\gamma}+\frac{n\omega}{2}\right)t,\quad e=0,1,\ldots,(n-1),\end{aligned} \tag{56}$$

and

$$|\Phi\rangle_{4e} = \exp\left[i\varphi_{4e}(t)\right]|\phi\rangle_{4e},$$
$$\varphi_{4e}(t) = -_{4e}\langle\Phi|\mathcal{H}|\Phi\rangle_{4e}\, t = -E_{4e}t \tag{57}$$
$$= -\left(\omega e - \frac{\varpi}{2\cos\gamma} + \frac{n\omega}{2}\right)t, \quad e = -n, \ldots, -2, -1.$$

Since $|\Phi\rangle_{3e}$ and $|\Phi\rangle_{4e}$ represent the states without the interaction between the thermo field and the atom, the total wavefunction $|\Psi(t)\rangle$ for the system and its evolution operator $U(t)$ are expressed as

$$|\Psi(t)\rangle = C_1|\Phi(t)\rangle_1 + C_2|\Phi(t)\rangle_2 \tag{58}$$

and

$$U(t) = e^{i\varphi_1(t)}|\phi\rangle_{11}\langle\phi| + e^{i\varphi_2(t)}|\phi\rangle_{22}\langle\phi| \tag{59}$$

where the coefficients $C_j = _j\langle\phi|\Psi(0)\rangle$, $(j = 1, 2)$ and $|\Psi(0)\rangle$ is an given initial state of the system.

Based on the general solution $|\Psi(t)\rangle$ and the time-development operator $U(t)$, further we are able to discuss the dynamical and statistics properties of physical quantities in the generalized TJC system for the different initial states $|\Psi(0)\rangle$ of the system.

For example, the time evolution of both the thermo excitation number and the atomic inversion operator can be calculated as, respectively

$$\langle h(t)\rangle = \sum_e\left[|C_1|^2\left(e + n\sin^2\frac{\gamma}{2}\right) + |C_2|^2\left(e + n\cos^2\frac{\gamma}{2}\right) + n\,\mathrm{Re}[C_1^* C_2 e^{i\varpi t/\cos\gamma}\sin\gamma]\right] \tag{60}$$

and

$$\langle\sigma_z(t)\rangle = \sum_e\left[\left(|C_1|^2 - |C_2|^2\right)\cos\gamma - 2\,\mathrm{Re}\left(C_1^* C_2 e^{i\varpi t/\cos\gamma}\sin\gamma\right)\right] \tag{61}$$

where

$$\sin\gamma = \frac{2\lambda\sqrt{\mathcal{N}_n}}{\sqrt{\varpi^2 + 4\lambda^2\mathcal{N}_n}}, \qquad \cos\gamma = \frac{\varpi}{\sqrt{\varpi^2 + 4\lambda^2\mathcal{N}_n}}. \tag{62}$$

Here the summation considers the relative weight for each value of e, for the Rabi oscillations in summation with different e have different frequencies.

When the initial state $|\Psi(0)\rangle = \|e, m\rangle \otimes |+\rangle \equiv \|e, m, +\rangle$, (60) and (61) are also simplified as

$$\langle h(t)\rangle = \sum_e\left[e + \frac{2\lambda^2\mathcal{N}_n n}{\varpi^2 + 4\lambda^2\mathcal{N}_n}\left(1 - \cos(\sqrt{\varpi^2 + 4\lambda^2\mathcal{N}_n}t)\right)\right] \tag{63}$$

and

$$\langle\sigma_z(t)\rangle = \sum_e\left[\frac{\varpi^2 + 4\lambda^2\mathcal{N}_n\cos(\sqrt{\varpi^2 + 4\lambda^2\mathcal{N}_n}t)}{\varpi^2 + 4\lambda^2\mathcal{N}_n}\right]. \tag{64}$$

5 Pseudo-Invariant Eigen-Operator of $\mathcal{H}$

In this section, we turn our attention to search for the pseudo-invariant eigen-operator O_e of the generalized TJC model. The so-called pseudo-invariant eigen-operator O_e should satisfy the following equation [21–24]

$$[[O_e, \mathcal{H}], \mathcal{H}] |\psi\rangle_i = (\Delta E)^2 O_e |\psi\rangle_i \tag{65}$$

in the eigenvector space $|\psi\rangle_i$ of conservation quantities of the dynamic system, where ΔE is the energy-level gap of the dynamic Hamiltonian.

From the Hamiltonian in (31), we can see that $\mathcal{H}$ is a linear combination of the operators N, σ_z, Q and $Q^\dagger$, hence we think that O_e for $\mathcal{H}$ may have the form

$$O_e = \mu \left(Q^\dagger + Q\right) + \nu\sigma_z, \tag{66}$$

where μ and ν are the two real constants to be determined. With the aid of the relations in (27)–(30), we calcutate that

$$[O_e, \mathcal{H}] = (\mu\varpi - 2\nu\lambda) \left(Q - Q^\dagger\right), \tag{67}$$

then

$$[[O_e, \mathcal{H}], \mathcal{H}] = (\mu\varpi - 2\nu\lambda) \left[\Gamma \left(Q^\dagger + Q\right) - 2\lambda N'\sigma_z\right]. \tag{68}$$

In view of the discussion of Sect. 4, substituting (68) into (65) and using (38) we readily obtain the following relation in the $\binom{\|e,m\rangle}{0}$ and $\binom{0}{\|e+n,m\rangle}$ space,

$$\mu = -\frac{\varpi}{2\lambda\mathcal{N}_n}\nu. \tag{69}$$

Thus in the Hilbert space spanned by the eigenstates of N', we can determine the expression of O_e

$$O_e = -\frac{\varpi}{2\lambda\mathcal{N}_n}\nu \left(Q^\dagger + Q\right) + \nu\sigma_z, \tag{70}$$

which is a found pseudo-invariant eigen-operator. Considering again (65) and (70), it is obtained that

$$\varpi^2 + 4\lambda^2\mathcal{N}_n = (\Delta E)^2, \tag{71}$$

so the energy-level gap for $\mathcal{H}$ in (31) is

$$\Delta E = \sqrt{\varpi^2 + 4\lambda^2\mathcal{N}_n}, \tag{72}$$

which is the same as the results in (54) and (55), i.e.,

$$E_2 - E_1 = \frac{\varpi}{\cos\gamma} = \sqrt{\varpi^2 + 4\lambda^2\mathcal{N}_n}. \tag{73}$$

6 Conclusions

In summary, by analyzing the characteristics of thermo field dynamics, we have presented the generalized thermo Jaynes-Cummings model at finite temperature including the thermal effects and introduce its physical explanation in more detail based on the thermal excitation representation. We also devote to solve the generalized TJC model via the Lewis-Riesenfeld invariant method, whose advantages are that it may treat the geometric phase factor, and obtain the explicit expressions, instead of the hidden form, for the evolution operator of the wavefunctions [26–28]. Its exact eigenenergy spectrum and the explicit expressions for the evolution operator of the wavefunctions are derived. Our analytical results are appealing. As a particular case, i.e., $n = 1$, the results recover to that of the Ref. [13]. In addition, we evaluate various expectation values of physical quantities, such as thermo excitation number and the atomic inversion operator. Finally, we also find the pseudo-invariant eigen-operator for the generalized TJC Hamiltonian, which directly leads to the energy-level gap of this system.

Acknowledgement This work was supported by the National Natural Science Foundation of China under grant numbers 10775097.

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